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Author(s)	Harada, Junichi
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Exact blow-up profile for a heat equation with a nonlinear boundary condition

Junichi Harada
The Education and Human Studies,
Akita University

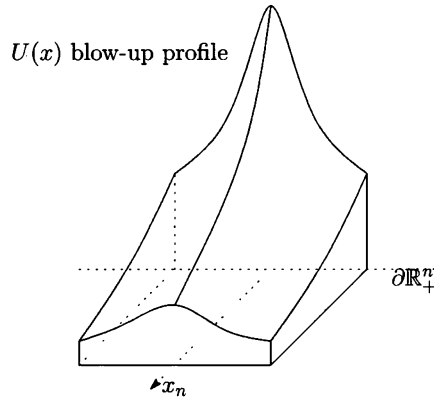
We study the asymptotic behavior of positive blow-up solutions for the heat equation.

$$\begin{cases} u_t = \Delta u, & x \in \mathbb{R}_+^n, \ t \in (0, T), \\ \partial_\nu u = u^q, & x \in \partial\mathbb{R}_+^n, \ t \in (0, T), \\ u = u_0, & x \in \mathbb{R}_+^n, \ t = 0, \end{cases} \quad (\text{P})$$

where $\mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n > 0\}$, $\partial_\nu = -\partial/\partial x_n$ and $1 < q < n/(n-2)$. A function $u(x, t)$ is said to be x_n -axial symmetric, if $u(x, t)$ depends only on $|x'|$ and x_n for any $t \in [0, T)$. We focus on x_n -axial symmetric functions throughout this report. Let $u(x, t)$ be a blow-up solution of (P) and T be its blow-up time. If a limit

$$U(x) = \lim_{t \rightarrow T} u(x, t) \in [0, \infty]$$

exists for any $x \in \mathbb{R}_+^n \cup \partial\mathbb{R}_+^n$, we call $U(x)$ a blow-up profile of $u(x, t)$.



A goal of this report is to determine the singularity of blow-up profiles. The blow-up profile for (P) was first studied in [4, 5] for a one dimensional case. Indeed they constructed blow-up solutions satisfying

$$U(x) = Ax^{\frac{-1}{q-1}} \quad \text{for } x \in \mathbb{R}_+. \quad (1)$$

The author [6] extended their results to a multidimensional case and proved that if $u(x, t)$ is a positive x_n -axial symmetric blow-up solution, its blow-up profile is given by

$$U(x) = A(1 + o(1))(\cos \theta)^{\frac{-1}{q-1}} |x|^{\frac{-1}{q-1}} \quad \text{along } x_n = |x| \cos \theta \quad (2)$$

for any fixed $\theta \in [0, \pi/2)$. This profile function coincides with (1) if $\theta = 0$. Unfortunately (2) does not hold on the boundary, since $\theta = \pi/2$ on the boundary. On the other hand, the boundary singularity was studied in [7]. He obtained the following inequalities under some monotonicity conditions.

$$k_1 (|\log |x||/|x|^2)^{\frac{1}{2(q-1)}} \leq U(x) \leq k_2 (|\log |x||/|x|^2)^{\frac{1}{2(q-1)}}, \quad x \in \partial\mathbb{R}_+^n, \ |x| < 1.$$

In this report, we improve this bound and consider more general situations.

Let $u(x, t)$ be a positive x_n -axial symmetric solution which blows up at the origin. We introduce self-similar variables.

$$w(y, s) = (T - t)^{\frac{1}{2(q-1)}} u(\sqrt{T - t} \cdot y, t), \quad T - t = e^{-s}.$$

The rescaled function $w(y, s)$ satisfies

$$\begin{cases} w_s = \Delta w - \frac{y}{2} \cdot \nabla w - \frac{w}{2(q-1)}, & y \in \mathbb{R}_+^n, \quad s > -\log T, \\ \partial_\nu w = w^q, & y \in \partial \mathbb{R}_+^n, \quad s > -\log T. \end{cases}$$

It is known that $w(y, s)$ converges to the $\kappa(y)$ as $s \rightarrow \infty$, where $\kappa(y) = \kappa(y_n)$ is the unique positive bounded solution of

$$\frac{d^2 \kappa}{dy_n^2} - \frac{y_n}{2} \frac{d\kappa}{dy_n} - \frac{\kappa}{2(q-1)} = 0 \quad (3)$$

(see [3], [2]). To investigate the asymptotic behavior of solutions, we consider the linearization around $\kappa(y)$.

$$\Delta \phi - \frac{y}{2} \cdot \nabla \phi - \frac{\phi}{2(q-1)} = -\lambda \phi \quad \text{in } \mathbb{R}_+^n, \quad \partial_\nu \phi = q\kappa^{q-1}\phi \quad \text{on } \partial \mathbb{R}_+^n. \quad (4)$$

Under x_n -axial symmetric case setting, the eigenfunction is written as

$$\phi_{k,l}(y) = h_k(|y'|) I_l(y_n), \quad k = 0, 1, 2, \dots, \quad l \in \mathbb{N}.$$

Let $\lambda_k^{\mathbb{R}^{n-1}}$ be the k th eigenvalue corresponding $h_k(|y'|)$ and $\lambda_l^{\mathbb{R}^+}$ be the l th eigenvalue corresponding $I_l(y_n)$. The eigenvalue of (4) is given by $\lambda_{k,l} = \lambda_k^{\mathbb{R}^{n-1}} + \lambda_l^{\mathbb{R}^+} + \frac{1}{2(q-1)}$. It is known that

$$\lambda_{0,1} < 0, \quad \lambda_{1,1} = 0, \quad \lambda_{k,l} > 0 \quad (k \geq 2 \text{ or } l \geq 2).$$

Let $L_\rho^2(\mathbb{R}_+^n)$ be a weighted Lebesgue space defined by $L_\rho^2(\mathbb{R}_+^n) = \{w \in L_{\text{loc}}^1(\mathbb{R}_+^n); \|w\|_{L_\rho^2(\mathbb{R}_+^n)} = (\int_{\mathbb{R}_+^n} w(y)^2 e^{-|y|^2/4} dy)^{1/2} < \infty\}$. We recall our previous result.

Theorem 1 (Theorem 1.1 [7]). *Let $u(x, t)$ be a positive x_n -axial symmetric solution of (P) which blows up at the origin. Then one of the following two cases occurs.*

- (c1) $\|w(s) - (\kappa + c_1 s^{-1} \phi_{1,1})\|_{L_\rho^2(\mathbb{R}_+^n)} = o(s^{-1})$, where $c_1 < 0$ is a constant depending only on n, q or
- (c2) $w(s) - \kappa$ decays exponentially in $L_\rho^2(\mathbb{R}_+^n)$.

We now state our main result in this report.

Theorem 2. *Let $u(x, t)$ be a positive x_n -axial symmetric solution of (P) which blows up at the origin.*

- (i) *If $w(y, s)$ behaves as (c1) in Theorem 1, then the blow-up profile $U(x) \in C(\mathbb{R}_+^n \cup \partial \mathbb{R}_+^n \setminus \{0\})$ exists and satisfies*

$$U(x) = k (|\log |x|| / |x|^2)^{\frac{1}{2(q-1)}} (1 + o(1)), \quad x \in \partial \mathbb{R}_+^n, \quad |x| < 1,$$

where k is a positive constant depending only on n and q .

- (ii) *If $w(y, s)$ behaves as (c2) in Theorem 1, then*

$$U(x) = k |x|^{-\frac{k}{(q-1)}} (1 + o(1)), \quad x \in \partial \mathbb{R}_+^n, \quad |x| < 1$$

for some $k \geq 2$.

Generally a blow-up solution behaves like

$$w(s) = \kappa + c_1 e^{-\lambda_k s} \phi_{k,l} + o(e^{-\lambda_k s}) \quad \text{in } L_\rho^2(\mathbb{R}_+^n)$$

for some $k = 0, 1, 2, \dots$ and $l \in \mathbb{N}$. Unfortunately we have no idea about the case $l \geq 2$.

Our proof is a slight modification of arguments in a series of papers [8, 9, 10, 11, 12]. We here focus on the asymptotic behavior of $w(y, s)$ for large $|y|$. We introduce another rescale.

$$W(z, \zeta, s) = w(R(s)z, \zeta, s) \quad \text{for } z \in \mathbb{R}^{n-1}, \zeta > 0.$$

The function $R(s)$ ($\lim_{s \rightarrow \infty} R(s) = \infty$) is determined by the long time behavior of $w(y, s)$. A goal is to determine the asymptotic behavior of $W(z, \zeta, s)$ for $|z| + |\zeta| < R$ as $s \rightarrow \infty$. We see that $W(z, \zeta, s)$ satisfies

$$\begin{cases} W_s = R^{-2} \Delta_z W - (\frac{1}{2} - \frac{R_s}{R}) z \cdot \nabla_z W + W_{\zeta\zeta} - \frac{\zeta}{2} W_\zeta - mW, & z \in \mathbb{R}^{n-1}, \zeta > 0, s > -\log T, \\ \partial_\nu W = W^q, & z \in \mathbb{R}^{n-1}, \zeta = 0, s > -\log T. \end{cases}$$

To investigate the long time behavior of $W(z, \zeta, s)$, we study a limiting problem of this equation. For the case (i), since $w(y, s) = \kappa(y) + c_1 s^{-1} \phi_{1,1}(y) + o(s^{-1})$ in $L^2_\rho(\mathbb{R}^n_+)$ (see Theorem 1 (c1)) and $\phi_{1,1}(y) = c(|y'|^2 - 2(n-1))I_1(y_n)$, we take

$$R(s) = \sqrt{s}.$$

A limiting equation is given by

$$\begin{cases} -\frac{1}{2} z \cdot \nabla_z W + W_{\zeta\zeta} - \frac{\zeta}{2} W_\zeta - mW = 0 & \text{for } z \in \mathbb{R}^{n-1}, \zeta > 0, \\ \partial_\nu W = W^2 & \text{on } z \in \mathbb{R}^{n-1}, \zeta = 0. \end{cases} \quad (5)$$

We will see that

$$W(z, \zeta) = (1 + b|z|^2)^{-m} \kappa\left(\frac{\zeta}{\sqrt{1+b|z|^2}}\right) \quad (6)$$

gives a solution of (5) (see below), where $b > 0$ is a free parameter. We choose b to match (c1) in Theorem 1. Then $W(z, \zeta, s)$ gives a better approximation of $w(y, s)$ than $\kappa + c_1 s^{-1} \phi_{1,1}$.

From now, we derive (6) in a formal approach. Since $z \cdot \nabla_z |z|^{2j} = 2j|z|^{2j}$, we look for solutions in the form of

$$W(z, \zeta) = \sum_{j=0}^{\infty} a_j(\zeta) |z|^{2j}.$$

Substituting this into (5), we have

$$a_j'' - \frac{\zeta}{2} a_j' - (m+j)a_j = 0, \quad \zeta > 0.$$

Let $A_j(\eta) = a_j(\zeta)$ with $\eta = \frac{\zeta^2}{4}$ for $\eta > 0$. Then A_j satisfies

$$\eta A_j'' + (\frac{1}{2} - \eta) A_j' - (m+j) A_j = 0, \quad \eta > 0.$$

This is the confluent hypergeometric equation (see (8.1.1) p. 201 [1]). One solution of this equation is give by the Kummer function $M(m+j, \frac{1}{2}; \eta)$, another is given by

$$U(m+j, \frac{1}{2}; \eta) = \frac{1}{\Gamma(m+j)} \int_0^\infty e^{-\eta t} t^{m+j-1} (1+t)^{\frac{1}{2}-(m+j)-1} dt$$

(see (8.2.1) p. 204 [1]). Since $M(m+j, \frac{1}{2}; \eta)$ behaves like $\eta^{m+j-\frac{1}{2}} e^\eta$ as $\eta \rightarrow \infty$ (see (8.1.8) p. 203 [1]), we take

$$A_j(\eta) = \alpha_j \pi^{-\frac{1}{2}} \Gamma(m+j+\frac{1}{2}) U(m+j, \frac{1}{2}, \eta),$$

where α_j is a constant. Since $U(a, c; 0) = \frac{\Gamma(1-c)}{\Gamma(a+1-c)}$ if $1-c > 0$ (see (8.2.5) p. 205 [1]), it is clear that

$$A_j(0) = \alpha_j. \quad (7)$$

Form now, we determine α_j . We substitute

$$W(z, \zeta) = \sum_{j=0}^{\infty} A_j \left(\frac{\zeta^2}{4}\right) |z|^{2j}$$

into a boundary condition in (5). Since $U'(a, c; x) = -aU(a+1, c+1, x)$ (see (8.5.8) p. 212 [1]) and $U(a, c; x) = \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c}(1+o(1))$ as $x \rightarrow 0$ if $1-c < 0$ (see p. 205 [1]), we get

$$\begin{aligned} \frac{d}{d\zeta} U(m+j, \tfrac{1}{2}, \tfrac{\zeta^2}{4})|_{\zeta=0} &= \tfrac{\zeta}{2} U'(m+j, \tfrac{1}{2}, \tfrac{\zeta^2}{4})|_{\zeta=0} \\ &= -\frac{(m+j)\zeta}{2} U(m+j+1, \tfrac{3}{2}, \tfrac{\zeta^2}{4})|_{\zeta=0} \\ &= -\frac{(m+j)\zeta}{2} \frac{\Gamma(\frac{1}{2})}{\Gamma(m+j+1)} \left(\frac{\zeta^2}{4}\right)^{-\frac{1}{2}} (1+o(1))|_{\zeta=0} \\ &= -(m+j) \frac{\Gamma(\frac{1}{2})}{\Gamma(m+j+1)} = -\frac{\Gamma(\frac{1}{2})}{\Gamma(m+j)}. \end{aligned}$$

Therefore it holds that

$$\frac{d}{d\zeta} A_j \left(\frac{\zeta^2}{4}\right)|_{\zeta=0} = -\alpha_j \pi^{-\frac{1}{2}} \Gamma(m+j+\tfrac{1}{2}) \frac{\Gamma(\frac{1}{2})}{\Gamma(m+j)} = -\alpha_j \frac{\Gamma(m+j+\frac{1}{2})}{\Gamma(m+j)}.$$

From now we consider the case $q = 2$. Since $m = \frac{1}{2(q-1)} = \frac{1}{2}$, it follows that

$$\partial_\nu W = -\partial_\zeta W|_{\zeta=0} = \sum_{j=0}^{\infty} \alpha_j \frac{\Gamma(m+j+\frac{1}{2})}{\Gamma(m+j)} |z|^{2j} = \sum_{j=0}^{\infty} \alpha_j \frac{\Gamma(1+j)}{\Gamma(\frac{1}{2}+j)} |z|^{2j}. \quad (8)$$

On the other hand, it holds from (7) that

$$\begin{aligned} W|_{\zeta=0}^2 &= (\alpha_0 + \alpha_1|z|^2 + \alpha_2|z|^4 + \alpha_3|z|^6 + \dots)(\alpha_0 + \alpha_1|z|^2 + \alpha_2|z|^4 + \alpha_3|z|^6 + \dots) \\ &= \alpha_0^2 + 2\alpha_0\alpha_1|z|^2 + (2\alpha_0\alpha_2 + \alpha_1^2)|z|^4 + (2\alpha_0\alpha_3 + 2\alpha_1\alpha_2)|z|^6 \\ &\quad + (2\alpha_0\alpha_4 + 2\alpha_1\alpha_3 + \alpha_2^2)|z|^8 + (2\alpha_0\alpha_5 + 2\alpha_1\alpha_4 + 2\alpha_2\alpha_3)|z|^{10} + \dots \end{aligned} \quad (9)$$

Therefore from (8) and (9), we obtain

$$\begin{aligned} \alpha_0 \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} &= \alpha_0^2, & \alpha_1 \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} &= 2\alpha_0\alpha_1, & \alpha_2 \frac{\Gamma(3)}{\Gamma(\frac{5}{2})} &= 2\alpha_0\alpha_2 + \alpha_1^2, \\ \alpha_3 \frac{\Gamma(4)}{\Gamma(\frac{7}{2})} &= 2\alpha_0\alpha_3 + 2\alpha_1\alpha_2, & \alpha_4 \frac{\Gamma(5)}{\Gamma(\frac{9}{2})} &= 2\alpha_0\alpha_4 + 2\alpha_1\alpha_3 + \alpha_2^2, \\ \alpha_5 \frac{\Gamma(5)}{\Gamma(\frac{9}{2})} &= 2\alpha_0\alpha_5 + 2\alpha_1\alpha_4 + 2\alpha_2\alpha_3. \end{aligned}$$

It is clear that $\alpha_0 = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} = \pi^{-\frac{1}{2}}$. For simplicity, we put $\alpha_j = \pi^{-\frac{1}{2}}\beta_j$. Then we get

$$\begin{aligned} \left(\frac{2!}{\frac{3}{2} \cdot \frac{1}{2}} - 2\right)\beta_2 &= \beta_1^2, & \left(\frac{3!}{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}} - 2\right)\beta_3 &= 2\beta_1\beta_2, & \left(\frac{4!}{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}} - 2\right)\beta_4 &= 2\beta_1\beta_3 + \beta_2^2, \\ \left(\frac{5!}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}} - 2\right)\beta_5 &= 2\beta_1\beta_4 + 2\beta_2\beta_3. \end{aligned}$$

Furthermore we put $\beta_j = (2\beta_1)^j \gamma_j$. We easily see that

$$\frac{8}{3}\gamma_2 = 1, \quad \frac{6}{5}\gamma_3 = \gamma_2, \quad \frac{58}{35}\gamma_4 = \gamma_3 + \gamma_2^2, \quad \frac{130}{63}\gamma_5 = \gamma_4 + 2\gamma_2\gamma_3.$$

This implies

$$\gamma_2 = \frac{3}{8} = \frac{\frac{1}{2} \cdot \frac{3}{2}}{2!}, \quad \gamma_3 = \frac{5}{16} = \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}}{3!}, \quad \gamma_4 = \frac{35}{128} = \frac{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}}{4!}, \quad \gamma_5 = \frac{63}{256} = \frac{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}}{5!}.$$

From this relation, we assume

$$\gamma_j = \frac{2j-1}{2} \frac{2j-3}{2} \dots \frac{3}{2} \frac{1}{2} = \pi^{-\frac{1}{2}} \frac{\Gamma(j+\frac{1}{2})}{j!}.$$

Since $\alpha_j = \pi^{-1}(2\beta_1)^j \gamma_j$, $W(z, \zeta)$ is written by

$$\begin{aligned} W(z, \zeta) &= \frac{\alpha_0}{\sqrt{\pi}} \Gamma(1) U(\tfrac{1}{2}, \tfrac{1}{2}; \eta) + \frac{\alpha_1}{\sqrt{\pi}} \Gamma(2) U(\tfrac{3}{2}, \tfrac{1}{2}; \eta) |z|^2 + \frac{1}{\sqrt{\pi}} \sum_{j=2}^{\infty} \alpha_j \Gamma(1+j) \\ &\quad \times U(\tfrac{1}{2} + j, \tfrac{1}{2}; \eta) |z|^{2j} \\ &= \frac{1}{\pi} U(\tfrac{1}{2}, \tfrac{1}{2}; \eta) + \frac{\beta_1}{\pi} U(\tfrac{3}{2}, \tfrac{1}{2}; \eta) |z|^2 + \frac{1}{\pi^{\frac{3}{2}}} \sum_{j=2}^{\infty} \Gamma(j + \tfrac{1}{2}) U(\tfrac{1}{2} + j, \tfrac{1}{2}; \eta) (2\beta_1)^j |z|^{2j} \\ &= \frac{1}{\pi^{\frac{3}{2}}} \sum_{j=0}^{\infty} \Gamma(j + \tfrac{1}{2}) U(\tfrac{1}{2} + j, \tfrac{1}{2}; \eta) (2\beta_1)^j |z|^{2j}. \end{aligned} \quad (10)$$

We recall the following formula (see p. 219 [1]).

$$U(a, \tfrac{1}{2}, \tfrac{\zeta^2}{4}) = \frac{\sqrt{\pi}}{\Gamma(a+\frac{1}{2})\Gamma(a)} \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma(a+\frac{l}{2})}{l!} \zeta^l. \quad (11)$$

We apply this formula in (10).

$$\begin{aligned} W(z, \zeta) &= \frac{1}{\pi^{\frac{3}{2}}} \sum_{j=0}^{\infty} \Gamma(j + \tfrac{1}{2}) U(\tfrac{1}{2} + j, \tfrac{1}{2}; \eta) (2\beta_1)^j |z|^{2j} \\ &= \frac{1}{\pi} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma(\frac{1}{2} + \frac{l}{2} + j)}{j! l!} (2\beta_1)^j |z|^{2j} \zeta^l. \end{aligned}$$

Since

$$(1-x)^{-p} = 1 + \frac{\Gamma(p+1)}{\Gamma(p)} x + \frac{\Gamma(p+2)}{2! \Gamma(p)} x^2 + \dots + \frac{\Gamma(p+n)}{n! \Gamma(p)} x^n + \dots,$$

we obtain

$$\begin{aligned} W(z, \zeta) &= \frac{1}{\pi} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(\frac{1}{2} + \frac{l}{2} + j)}{j! l!} (2\beta_1)^j |z|^{2j} \zeta^l \\ &= \frac{1}{\pi} \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma(\frac{1}{2} + \frac{l}{2})}{l!} \zeta^l \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2} + \frac{l}{2} + j)}{j! \Gamma(\frac{1}{2} + \frac{l}{2})} (2\beta_1 |z|^2)^j \\ &= \frac{1}{\pi} \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma(\frac{1}{2} + \frac{l}{2})}{l!} \zeta^l (1 - 2\beta_1 |z|^2)^{-(\frac{1}{2} + \frac{l}{2})} \\ &= \frac{1}{\pi} (1 - 2\beta_1 |z|^2)^{-\frac{1}{2}} \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma(\frac{1}{2} + \frac{l}{2})}{l!} \left(\frac{\zeta}{\sqrt{1 - 2\beta_1 |z|^2}} \right)^l. \end{aligned}$$

We set $\eta_1 = \frac{\zeta^2}{4(1-2\beta_1|z|^2)}$. We again use (11) to get

$$\begin{aligned} W(z, \zeta) &= \frac{1}{\pi} (1 - 2\beta_1 |z|^2)^{-\frac{1}{2}} \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma(\frac{1}{2} + \frac{l}{2})}{l!} \left(\frac{\zeta}{\sqrt{1 - 2\beta_1 |z|^2}} \right)^l \\ &= \frac{1}{\pi} (1 - 2\beta_1 |z|^2)^{-\frac{1}{2}} U(\tfrac{1}{2}, \tfrac{1}{2}; \eta_1). \end{aligned}$$

We put $-2\beta_1 = \mathbf{b}$. We finally obtain

$$W(z, \zeta) = \frac{1}{\pi} (1 + \mathbf{b}|z|^2)^{-\frac{1}{2}} U(\tfrac{1}{2}, \tfrac{1}{2}; \tilde{\eta}_1), \quad \tilde{\eta}_1 = \frac{\zeta^2}{4(1+\mathbf{b}|z|^2)}.$$

Let $\kappa(\zeta)$ be given in (3). Since $\kappa(\zeta) = \frac{1}{\pi}U(\frac{1}{2}, \frac{1}{2}, \frac{\zeta^2}{4})$ (see (3.6) in [5]), we obtain

$$W(z, \zeta) = (1 + b|z|^2)^{-\frac{1}{2}} \kappa\left(\frac{\zeta}{\sqrt{1+b|z|^2}}\right).$$

By a direct computation, we can check this actually satisfies (5) with $q = 2$. By the same way, we find that

$$W(z, \zeta) = (1 + b|z|^2)^{-m} \kappa\left(\frac{\zeta}{\sqrt{1+b|z|^2}}\right).$$

gives a solution of (5) for any $q > 1$.

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